Classical and Quantum Nonultralocal Systems on the Lattice

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Abstract

We classify nonultralocal Poisson brackets for 1-dimensional lattice systems and describe the corresponding regularizations of the Poisson bracket relations for the monodromy matrix . A nonultralocal quantum algebras on the lattices for these systems are constructed.For some class of such algebras an ultralocalization procedure is proposed.The technique of the modified Bethe-Anzatz for these algebras is developed.This technique is applied to the nonlinear sigma model problem.

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Introduction

This article is devoted to an old problem, which arose in the beginning of the development of the Classical Inverse Scattering Method (CISM) [1]. An important point of CISM is the calculation of the Poisson brackets relations for the monodromy matrix of an auxiliary linear problem. This calculation is usually performed under the technical assumption of 'ultralocality' of the Poisson brackets for local variables (this condition means simply that the Poisson operator defining the bracket is a multiplication operator and does not contain any derivations). In many interesting models this condition is violated, and in this case getting consistent Poisson brackets relations for the monodromy becomes nontrivial. Technically, the trouble is that the Frechet derivative of the monodromy has a discontinuity, and so one has to extend a differential operator to functions with a jump. It is easy to observe that Poisson operators are nonultralocal precisely for the models with non-skew-symmetric r-matrices. A naive calculation of the Poisson brackets for the monodromy in this case gives:

$$\begin{aligned} \{M_1, M_2\} &= aM_1M_2 - M_1M_2a, \\ a &= \tfrac{1}{2}\left(r - r^*\right). \end{aligned} \tag{1}$$

This bracket does not satisfy the Jacobi identity, since the skew part of r usually does not satisfy the Yang-Baxter identity (in fact, the bracket(1) is inconsistent even if it does). A natural way to regularize the monodromy brackets in this case has been proposed in [19]. This method allows to regularize some (though not all) of the Poisson brackets of the type (1). The idea is that to extend the Poisson operator to functions with a jump one has to add to it a boundary form sensitive to the jump, which is well in the spirit of the operator extensions theory. In this article we classify all regularized r-matrices and all regularizations of this kind using the Belavin-Drinfeld classification theorem for the modified Yang-Baxter equation [5]. Unfortunately, our classification is given in an implicit form because the Belavin-Drinfeld classification theorem describes solutions of the modified Yang-Baxter equation only up to automorphisms of the corresponding affine Lie algebra. This fact doesn't enable to write all regularizations in the explicit form. But we give a natural way to find all regularizations. We define the corresponding quantum algebras by means of the Faddeev-Reshetikhin-Takhtajan approach [10]. The same class of Poisson structures and of the corresponding quantum

algebras has been recently studied in a slightly different way by J.M.Maillet and L.Freidel [22] and by S.Parmentier [21]; to describe them we use the unified approach based on the notion of the twisted double (cf.[18], [20])

The second goal of the present work is to construct quantum nonultralocal systems on the lattice ,which possess infinite series of conservation laws and to calculate the spectrum of the corresponding commuting operators. For this calculation we develop a generalization of the Bethe-Ansatz construction.

Some words about the contents of this paper.

In section 1 we review the construction of Poisson algebras on the lattice arising in the study of Lax equations on the lattice with non-ultralocal Poisson brackets.

In section 2 we remind the main construction of [19]. We reformulate the Belavin-Drinfeld classification theorem [5] in terms of the affine root systems. This reformulation is convenient for our purposes. We reduce the classification of regularizations to the search of some class of solutions of the Yang-Baxter equation on the square of a finite-dimensional Lie algebra.

In section 3 we discuss the main examples of regularizations and the corresponding nonultralocal algebras and investigate their algebraic properties. In particular, we determine their centers; under some additional conditions it is possible to find a new system of generators of these algebras which already satisfy *local* commutation relations This ultralocalization procedure has been discussed earlier in [13]. We present new examples of ultralocalization; the new system of generators is related to the original one by an appropriate quantum lattice gauge transformation. At the end of section 3 we describe a generalization of the algebraic Bethe Ansatz for nonultralocal algebras.

In section 4 we apply the technique developed in the previous sections to the nonlinear sigma model problem. It is well known that integrable models usually admit several different Poisson structures; the simplest one for the nonlinear sigma model is associated with its standard Lagrangian formulation. We were unable to find a regularization of this Poisson structure; however, the general scheme introduced in section 2 may be applied to another, and a fairly natural Poisson structure which we introduce in this section for a nonlinear sigma model with values in an arbitrary Riemannian symmetric space. We explicitly describe the corresponding quantum lattice systems . For the n-field (i.e., the sigma model with values in the unit sphere S^2) we get a representation of the local quantum lattice Lax operator via the canonical Weyl pairs. It turns out that the n-field with this Poisson structure

is gauge equivalent to the lattice Sine-Gordon model.

In the conclusion we discuss some open problems.

1 General construction of lattice algebras

It is natural to assume that the phase space of a mechanical system associated with a 1-dimensional lattice $\Gamma = \mathbf{Z}/N\mathbf{Z}$ is the direct product $\mathcal{M}^N$ of "1-particle spaces". In applications to integrable systems these "elementary" phase spaces are parametrized by Lax matrices and hence are modeled on submanifolds of an appropriate Lie group (usually, a loop group associated with a finite-dimensional semisimple Lie group). In simple cases the Poisson structure on $\mathcal{M}^N$ is the product structure. (The corresponding Poisson bracket is called *ultralocal*.) The auxiliary linear problem associated with Lax equations on the lattice is

$$\psi_{n+1} = L^n \psi_n. \tag{2}$$

The associated monodromy map is the product map

$$M : G^N \to G : (L^1, ..., L^N) \mapsto \prod_{n=1}^{N} L^n. \tag{3}$$

It is natural to demand that M is a Poisson map. In ultralocal case this condition means that G should be a Poisson Lie group. It is interesting (and also important for applications) to study the most general Poisson structures on G^N which are compatible with this property of the monodromy. The corresponding Poisson algebras are referred to as *lattice algebras*. First examples of nonultralocal lattice algebras appeared in [18]; further examples and a classification (for finite dimensional semisimple Lie algebras) appeared in [22], [4], [21]. In this section we briefly recall the construction of lattice algebras using the approach proposed in [18], [19].

Fix an affine Lie algebra $\mathbf{g}$ with a normalized invariant bilinear form $\langle \cdot, \cdot \rangle$. It is well known that $\mathbf{g}$ admits the structure of a quasitriangular Lie bialgebra (the corresponding classical r-matrices are listed in [5]). Put $\mathbf{d} = \mathbf{g} \oplus \mathbf{g}$. We define the bilinear invariant form on the square of $\mathbf{g}$ in the following way:

$$\langle\langle (X_1, Y_1), (X_2, Y_2) \rangle\rangle = \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle, \tag{4}$$

so that the diagonal subalgebra is isotropic. As a Lie algebra, $\mathbf{d}$ is isomorphic to the double of $\mathbf{g}$. (This isomorphism does not depend on a particular choice of the r-matrix.) Hence $\mathbf{d}$ carries a natural r-matrix, the r-matrix of the double; for our present goals, however, we shall need *arbitrary* classical r-matrices on $\mathbf{d}$ which define on it the structure of a quasitriangular Lie bialgebra. In other words, we are interested in r-matrices which are skew with respect to(4) and satisfy the modified classical Yang-Baxter equation on $\mathbf{g} \oplus \mathbf{g}$.

Let $\mathcal{R}$ be such a solution; it may be written in the block form:

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \ A^* = -A, \ D^* = -D. \tag{5}$$

For $\varphi \in Fun\,(D)\,, D = G \times G$ let $\mathrm{D}\varphi, \ \mathrm{D}'\varphi \in (\mathbf{g} \oplus \mathbf{g})^*$ be the left and right derivatives of φ:

$$\begin{aligned}
\langle\langle \mathrm{D}\varphi\,(g)\,, X\rangle\rangle &= \tfrac{d}{dt}\mid_{t=0} \varphi\left(e^{tX}g\right), \\
\langle\langle \mathrm{D}'\varphi\,(g)\,, X\rangle\rangle &= \tfrac{d}{dt}\mid_{t=0} \varphi\left(ge^{tX}\right), \\
g &\in D, \ X \in \mathbf{g} \oplus \mathbf{g}.
\end{aligned} \tag{6}$$

It is well known that for any two solutions $\mathcal{R}, \mathcal{R}'$ of the modified classical Yang-Baxter equation the bracket

$$\{\varphi, \psi\}_{\mathcal{R},\mathcal{R}'} = \langle\langle \mathcal{R}_1 \mathrm{D}\varphi, \mathrm{D}\psi\rangle\rangle + \langle\langle \mathcal{R}_2 \mathrm{D}'\varphi, \mathrm{D}'\psi\rangle\rangle \tag{7}$$

satisfies the Jacobi identity. Let us take, in particular, $\mathcal{R}_1 = \mathcal{R}, \mathcal{R}_2 = \pm\mathcal{R}$ we get the following important brackets

$$\{\varphi, \psi\}_{D_\pm} = \langle\langle \mathcal{R}\mathrm{D}\varphi, \mathrm{D}\psi\rangle\rangle \pm \langle\langle \mathcal{R}\mathrm{D}'\varphi, \mathrm{D}'\psi\rangle\rangle\,. \tag{8}$$

We denote by $D_\pm$ the group D with the bracket $\{\cdot, \cdot\}_{D_\pm}$. The bracket $\{,\}_{D_-}$ equips D with the structure of a Poisson-Lie group, while the "+" sign corresponds to an almost nondegenerate Poisson structure on D_+. (It is symplectic on an open cell in D containing the unit element, see [23] for the description of the symplectic leaves of D_+.)

Multiplication map $D \times D \to D$ defines a Poisson group action $D_- \times D_+ \to D_+$; its restriction to the diagonal subgroup $G \subset D$ is *admissible* [18], and hence it is possible to perform Poisson reduction over the action of G. The quotient space is canonically identified with G itself; in fact, it is clear that

the map $\pi : D \to G : (g_1, g_2) \mapsto g_1 g_2^{-1}$ is constant on the right coset classes of G.

To calculate the explicit form of the quotient Poisson structure on G choose $\varphi \in Fun\,(G)$ and put $\hat{\varphi} = \pi^* \varphi$; let $\nabla \varphi, \nabla' \varphi$ be the left and right derivatives of φ.

$$
\begin{aligned}
\langle \nabla \varphi\,(g)\,, X \rangle &= \tfrac{d}{dt}\,|_{t=0}\,\varphi \left(e^{tX} g \right), \\
\langle \nabla' \varphi\,(g)\,, X \rangle &= \tfrac{d}{dt}\,|_{t=0}\,\varphi \left(g e^{tX} \right), \\
g &\in G,\; X \in \mathbf{g}.
\end{aligned}
\tag{9}
$$

Then
$$
\mathrm{D}\hat{\varphi}\,(g_1, g_2) = \left(\nabla \varphi \left(g_1 g_2^{-1} \right), \nabla' \varphi \left(g_1 g_2^{-1} \right) \right).
\tag{10}
$$

After a short computation this yields:

$$
\{\varphi, \psi\}_G = \langle A \nabla \varphi, \nabla \psi \rangle - \langle D \nabla' \varphi, \nabla' \psi \rangle + \langle B \nabla' \varphi, \nabla \psi \rangle - \langle B^* \nabla \varphi, \nabla' \psi \rangle.
\tag{11}
$$

In general, this Poisson structure is degenerate.

Suppose that τ is an automorphism of $\mathbf{g}$;then $\tau \oplus \tau$ is an automorphism of $\mathbf{g} \oplus \mathbf{g}$. Let us assume that $\tau \oplus \tau$ commutes with $\mathcal{R}$. To twist the r-matrix on $\mathbf{d}$ we shall use another extension of τ to $\mathbf{d}$; namely, we put $\hat{\tau} = \tau \oplus \tau^{-1}$. Put

$$
\mathcal{R}^\tau = \hat{\tau} \mathcal{R} \hat{\tau}^{-1} = \begin{pmatrix} A & B\tau^{-1} \\ \tau B^* & D \end{pmatrix}.
\tag{12}
$$

$\mathcal{R}^{\;\tau}$ satisfies the Yang-Baxter equation. Put $\mathcal{R}_1 = \mathcal{R}^\tau, \mathcal{R}_2 = \mathcal{R}$ in (7) and denote by D_τ the group D with the corresponding Poisson structure:

$$
\{\varphi, \psi\}_{D_\tau} = \langle\langle \mathcal{R}^\tau \mathrm{D}\varphi, \mathrm{D}\psi \rangle\rangle + \langle\langle \mathcal{R} \mathrm{D}'\varphi, \mathrm{D}'\psi \rangle\rangle.
\tag{13}
$$

(If $\mathcal{R}$ is the r-matrix of the double of $\mathbf{g}$, the group D_τ is usually referred to as the *twisted double*.) This Poisson structure on D_τ also admits reduction with respect to the action of the diagonal subgroup; the quotient structure on $G = \pi(D_\tau)$ is given by

$$
\{\varphi, \psi\}_\tau = \langle A \nabla \varphi, \nabla \psi \rangle - \langle D \nabla' \varphi, \nabla' \psi \rangle + \left\langle B\tau^{-1} \nabla' \varphi, \nabla \psi \right\rangle - \langle \tau B^* \nabla \varphi, \nabla' \psi \rangle.
\tag{14}
$$

In particular, let us apply this construction to the group $\mathsf{G} = G^N$;in this case $\mathsf{D} = (G \times G)^N$, and τ is the cyclic shift in the direct sum $\bigoplus_1^N \mathbf{g}$. Let

$$r = \left(\begin{array}{cc} A & B \\ B^* & D \end{array} \right) \qquad (15)$$

be a solution of the modified classical Yang-Baxter equation on $\mathbf{g} \oplus \mathbf{g}$; put $\mathcal{R} = \bigoplus_1^N r$. Evidently, $\mathcal{R}$ commutes with $\tau \oplus \tau$. To describe the resulting lattice Poisson algebra it is convenient to introduce tensor notations. Fix an exact matrix representation ρ_V of G and denote

$$\begin{array}{c} L^n = \rho_V \left(g_n \right), L_1^n = L^n \otimes I, L_2^n = I \otimes L^n, \\ g = (g_1,\dots,g_n) \in G^N. \end{array} \qquad (16)$$

The reduced Poisson brackets of the matrix coefficients of L^n have the form:

$$\begin{array}{c} \{L_1^n, L_2^n\} = -A L_1^n L_2^n + L_1^n L_2^n D, \\ \left\{ L_1^n, L_2^{n+1} \right\} = L_1^n B^* L_2^{n+1}, \\ \{L_1^n, L_2^m\} = 0, |n - m| \geq 2. \end{array} \qquad (17)$$

Here we denote $(\rho_V \otimes \rho_V) A$ as well as A ,etc. The main property of the Poisson bracket (17) is given by the following assertion:

Theorem 1.1 *[20]*

Equip $\mathsf{G} = \mathrm{G}^N$ with the Poisson structure (11); then the monodromy map

$$M : \mathsf{G}_\tau \to \mathsf{G}, \ M \left(g_1, \dots, g_N \right) = g_1 \cdot \dots \cdot g_N$$

is a Poisson mapping if and only if the r-matrix (15) satisfies the additional constraint

$$A + B = B^* + D. \qquad (18)$$

In that case the Poisson structure in the target space of the monodromy map is given by (11).

In tensor notations we have the following brackets for M :

$$\{M_1, M_2\} = -AM_1M_2 + M_1M_2D + M_1B^*M_2 - M_2BM_1 \qquad (19)$$

Later we shall describe symplectic leaves of the bracket (17) in the main examples. In the particular case when $B = 0, A = D$ the bracket is ultralocal. The reader may keep in mind this possibility as a degenerate case.

The remainder of this section is devoted to the quantization of the Poisson brackets (17), (19). It may be easily performed on the lines of [10] provided that we know the quantum R-matrix which corresponds to the chosen classical r-matrix on $\mathbf{d}$. More precisely, let $U_q(\mathbf{d};\mathcal{R})$ be the quantized universal enveloping algebra of $\mathbf{d}$ which corresponds to $\mathcal{R}$ [6]; note that its description is not quite obvious since in the existing literature only the standard algebras $U_q(\mathbf{d};\mathcal{R})$ which correspond to simplest solutions of the classical Yang-Baxter equation are usually considered. It is widely believed that all solutions from the Belavin-Drinfeld list [5] give rise to quasitriangular Hopf algebras. (Examples in section 3 below give evidence to support this belief.) Assuming that the algebra $U_q(\mathbf{d};\mathcal{R})$ exists, let

$$\mathcal{R}_q = \begin{pmatrix} A_q & B_q \\ B_q^* & D_q \end{pmatrix} \in U_q(\mathbf{d};\mathcal{R}) \otimes U_q(\mathbf{d};\mathcal{R}) \qquad (20)$$

be its universal quantum R-matrix. We omit the explicit form of the relations in the algebra $U_q(\mathbf{d};R)$. Let ρ_V be some representation of the algebra $U_q(\mathbf{d};\mathcal{R})$ in the space V and let

$$\mathcal{R}_q^{VV} = (\rho_V \otimes \rho_V)\mathcal{R}_q = \begin{pmatrix} A_q & B_q \\ B_q^* & D_q \end{pmatrix} \qquad (21)$$

The following theorem is parallel to the description of the twisted quantum double and of the lattice current algebra [20], [4].

Theorem 1.2 *The free algebra $Fun_q^{\mathcal{R}}(\mathsf{G}_\tau)$ generated by the matrix elements of the matrices $L^n \in Fun_q^{\mathcal{R}}(\mathsf{G}_\tau) \otimes End(V)$, satisfying the following relations:*

$$\begin{aligned} A_q L_1^n L_2^n &= L_2^n L_1^n D_q \\ L_1^n B_q^{*-1} L_2^{n+1} &= L_2^{n+1} L_1^n \end{aligned}, \qquad (22)$$

is the quantization of the Poisson algebra (17).

Theorem 1.3 *The free algebra $Fun_q^{\mathcal{R}}(\mathsf{G})$ generated by the matrix elements of the matrix $M \in Fun_q^{\mathcal{R}}(\mathsf{G}) \otimes End(V)$, satisfying the relations:*

$$A_q M_1 {B_q^*}^{-1} M_2 = M_2 B_q^{-1} M_1 D_q \qquad (23)$$

is the quantization of the Poisson algebra (19).

Finally, we formulate the quantum version of theorem 1.1

Theorem 1.4 *The map*

$$M : Fun_q^{\mathcal{R}}(\mathsf{G}_\tau) \to Fun_q^{\mathcal{R}}(\mathsf{G}),$$
$$\left(L^1, \ldots, L^N\right) \mapsto L^1 \cdot \ldots \cdot L^N$$

is an homomorphism of algebras.

The algebras (22), (23) are the principal objects of our investigation.

2 Regularization of nonultralocal Poisson brackets

The goal of this section is to link the construction of lattice algebras with Hamiltonian systems on coadjoint orbits of current algebras. This approach is outlined in [19] where a regularization procedure for the Poisson brackets of the monodromy matrix is proposed which matches naturally with lattice Poisson algebras described in section 1. This will also allow us to construct consistent lattice approximations of nonultralocal systems on the circle.

We remind some details of the construction of dynamical systems on coadjoint orbits [17], [1].

Let $\mathbf{G} = C^\infty(S^1, \mathbf{g})$ be a current algebra with the values in some affine Lie algebra $\mathbf{g}$. Let us define the invariant scalar product on $\mathbf{G}$:

$$(X, Y) = \int\limits_0^{2\pi} \langle X, Y \rangle \, dz, \qquad (24)$$

where $X, Y \in \mathbf{G}, \langle \cdot, \cdot \rangle$ is a invariant bilinear form on $\mathbf{g}$. Let $\widehat{\mathbf{G}}$ be the central extension of the algebra $\mathbf{G}$ which corresponds to the 2-cocycle

$$\omega(X, Y) = (X, \partial_z Y). \tag{25}$$

By definition, $\widehat{\mathbf{G}}$ is the set of pairs $(X, a), X \in \mathbf{G}, a \in \mathbf{C}$ with the commutator

$$[(X.a), (Y, b)] = ([X, Y], \omega(X, Y)). \tag{26}$$

If r is a solution of the modified classical Yang-Baxter equation on $\mathbf{g}$, we put as usual

$$[X, Y]_r = [rX, Y] + [X, rY].$$

Let $\mathbf{g}_r$ be the algebra $\mathbf{g}$ equipped with this bracket. Put $\mathbf{G}_r = C^\infty(S^1, \mathbf{g}_r)$; it is easy to see that

$$\omega_r(X, Y) = \omega(rX, Y) + \omega(X, rY)$$

is a 2-cocycle on $\mathbf{G}_r$; thus we may define the second structure of a Lie algebra on $\widehat{\mathbf{G}}$

$$[(X, a), (Y, b)]_r = ([X, Y]_r, \omega_r(X, Y)). \tag{27}$$

In this formula it is not assumed that r is skew-symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle$. (In fact, if it is, the cocycle ω_r vanishes identically.)

Let $\widehat{\mathbf{G}}^*$ be the dual space of $\widehat{\mathbf{G}}$; using the inner product (24) we may identify it with $\mathbf{G} \oplus \mathbf{C}$. The Poisson bracket used in the CISM is the Lie-Poisson bracket which corresponds to the commutator (27). The variable $e \in \mathbf{C}$ is central with respect to this bracket. If $X_\varphi \in \widehat{\mathbf{G}}$ is a derivative of a function $\varphi \in Fun\left(\widehat{\mathbf{G}}^*\right)$:

$$((X_\varphi, X)) = \frac{d}{dt}\big|_{t=0} \varphi(L + tX),$$
$$X, L \in \widehat{\mathbf{G}}^*, \tag{28}$$
here $((\cdot, \cdot))$ is a natural pairing between $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{G}}^*$.

then

$$\{\varphi, \psi\}(L, e) = \left(\left((L, e), [X_\varphi, X_\psi]_r\right)\right). \tag{29}$$

Without loss of generality we may assume that $e = 1$ and suppress it in the notations. The bracket (29) may be represented as the bilinear form of the Poisson operator:

$$\mathcal{H} = adL \circ r + r^* \circ adL - (r + r^*) \, \partial_z, \tag{30}$$

$$\{\varphi, \psi\} (L) = (\mathcal{H}X_\varphi, X_\psi). \tag{31}$$

The operator $\mathcal{H}$ is unbounded, so the formula (31) requires some caution when the gradients are not smooth on the circle. This is precisely the case for the Poisson brackets of the monodromy matrix. Let ψ be the fundamental solution of the equation:

$$\partial_z \psi = L\psi \tag{32}$$

normalized by $\psi(0) = I$; then the monodromy matrix is equal to

$$M = \psi (2\pi) \in G. \tag{33}$$

Fix $\Phi \in Fun(G)$. According to [1], the Frechet derivative of the functional $L \mapsto \Phi (M [L])$ is given by

$$X_\Phi (z) = \psi (z) \, \nabla' \Phi (M) \, \psi (z)^{-1} \tag{34}$$

and in general is discontinuous at $z = 0$:

$$\begin{aligned} X_\Phi (0) &= \nabla' \Phi (M), \\ X_\Phi (2\pi) &= \nabla \Phi (M). \end{aligned} \tag{35}$$

To regularize Poisson brackets of the monodromy we shall use an idea borrowed from the theory of self-adjoint extensions [19]. Let $\triangle : C^\infty([0, 2\pi] \, ; \mathbf{g}) \to \mathbf{g} \oplus \mathbf{g}$ be the map which associates to a function on $[0, 2\pi]$ its boundary values,

$$\triangle X_\varphi = \begin{pmatrix} X_\varphi (0) \\ X_\varphi (2\pi) \end{pmatrix}. \tag{36}$$

Choose $B \in End \left(\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}} \right)$, here $\overset{\circ}{\mathbf{g}} \subset \mathbf{g}$ is the spreading finite-dimensional Lie algebra which corresponds to an affine Lie algebra $\mathbf{g}$; we extend operator B to the space $\mathbf{g} \oplus \mathbf{g}$ as a zero operator outside $\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$ and define the regularized Poisson bracket in the following way:

$$\{\varphi, \psi\} (L, 1) = \frac{1}{2} ((\mathcal{H}X_\varphi, X_\psi) - (\mathcal{H}X_\psi, X_\varphi)) + \langle\langle B\triangle X_\varphi, \triangle X_\psi \rangle\rangle. \tag{37}$$

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The bracket (37) must coincide with the bracket (31) on smooth functions, hence B must satisfy the condition:

$$\left\langle\left\langle B\begin{pmatrix} X \\ X \end{pmatrix}, \begin{pmatrix} Y \\ Y \end{pmatrix}\right\rangle\right\rangle = 0. \tag{38}$$

The additional restriction on B imposed in [19] follows from the study of the linearized bracket for the monodromy (37) for $M \to 1$; it is natural to demand that this linearized bracket should coincide with the one defined by r. This gives, after a short computation:

$$\{\Phi, \Psi\}(M) = \left\langle\left\langle \mathcal{R}\begin{pmatrix} \nabla\Phi \\ \nabla'\Phi \end{pmatrix}, \begin{pmatrix} \nabla\Psi \\ \nabla'\Psi \end{pmatrix}\right\rangle\right\rangle, \tag{39}$$

where

$$\mathcal{R} = \begin{pmatrix} -a+\alpha & -\alpha-s \\ \alpha-s & -a-\alpha \end{pmatrix}, \alpha^* = -\alpha, a = \tfrac{1}{2}(r-r^*), s = \tfrac{1}{2}(r+r^*),$$

$$\text{where } \alpha \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}} \text{ because } B \in End\left(\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}\right), \tag{40}$$

and our choice of B supposes that $s \in \overset{\circ}{\mathbf{g}} \otimes \overset{\circ}{\mathbf{g}}$. The Jacobi identity for this bracket will be valid if $\mathcal{R}$ satisfies the modified Yang-Baxter equation. In tensor notations a Poisson brackets of monodromy matrix have the form:

$$\{M_1, M_2\} = (a-\alpha)M_1M_2 - M_1M_2(a+\alpha) + M_1(\alpha-s)M_2 + M_2(\alpha+s)M_1. \tag{41}$$

The corresponding lattice Poisson algebra for which the monodromy matrix has the brackets (41) is:

$$\begin{array}{c} \{L_1^n, L_2^n\} = (a-\alpha)L_1^nL_2^n - L_1^nL_2^n(a+\alpha), \\ \left\{L_1^n, L_2^{n+1}\right\} = L_1^n(\alpha-s)L_2^{n+1}, \\ \{L_1^n, L_2^m\} = 0, |n-m| \geq 2. \end{array} \tag{42}$$

Our next step is a classification of the Poisson brackets of type (31) for which the Poisson brackets of the monodromy matrix may be regularized. It is difficult to classify all non-skew solutions of the Yang-Baxter equation for which there exists an $\alpha \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}}$ such that a matrix $\mathcal{R}$ in (40) is a solution of

the Yang-Baxter equation. But we can easily construct all solutions of the Yang-Baxter equation for $\mathbf{g} \oplus \mathbf{g}$ according to the Belavin-Drinfeld classification theorem [5], and then choose solutions of the form (40). To realize this program we start with an easy theorem:

Theorem 2.1 *If $\mathcal{R}$ is a solution of the modified Yang-Baxter equation for $\mathbf{g} \oplus \mathbf{g}$ of the type (40), then $a + s$ is a solution of the modified Yang-Baxter equation for $\mathbf{g}$.*

Proof. Let $r_1 = -a + \alpha, r_2 = -\alpha - s$, then $r = -(r_1 + r_2)$. From the Yang-Baxter equation for $\mathcal{R}$ it follows:

$$
\begin{aligned}
[r_1 X, r_2 Y] - r_1 \left([r_1 X, Y] + [X, r_1 Y]\right) &= -[X, Y], \\
[r_2 X, r_2 Y] - r_2 \left([(r_1 - 2\alpha) X, Y] + [X, (r_1 - 2\alpha) Y]\right) &= 0, \\
[r_1 X, r_2 Y] - r_1 [X, r_2 Y] - r_2 [(r_2 + 2\alpha) X, Y] &= 0, \\
X, Y &\in \mathbf{g}.
\end{aligned}
\tag{43}
$$

It is easy to check that the (43) implies the Yang-Baxter equation for r:

$$
[rX, rY] - r \left([rX, Y] + [X, rY]\right) = -[X, Y].
\tag{44}
$$

Let us turn now to the detailed study of the case of affine Lie algebras. We use the terminology and notations of the book [2].

Let $\mathbf{g}$ be an affine Lie algebra, $\triangle_+$ the set of its positive roots, $\overset{\circ}{\mathbf{g}}$ the corresponding spreading simple Lie algebra, $\overset{\circ}{\triangle}_+$ the set of its positive roots. Let $\triangle_{++} = \triangle_+ \backslash \overset{\circ}{\triangle}_+$. Using this notations we formulate some version of the Belavin-Drinfeld classification theorem.

Theorem 2.2 *[5] Up to an automorphism any solution of the modified classical Yang-Baxter equation for an affine Lie algebra $\mathbf{g}$ has the form:*

$$
R = \sum_{\alpha \in \triangle_{++}} e_\alpha \wedge e_{-\alpha} + r,
$$

where r is a solution of the modified Yang-Baxter equation for $\overset{\circ}{\mathbf{g}}$.

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For an explicit form of such solutions see [5].

In [5] such solutions are called trigonometric.

Thus from theorem 2.2 we have the following ansatz for a:

$$a = \sum_{\alpha \in \triangle_{++}} e_\alpha \wedge e_{-\alpha} + a_0, a_0 \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}}. \qquad (45)$$

and we reduce our problem to the Yang-Baxter equation on the square of a finite dimensional Lie algebra $\overset{\circ}{\mathbf{g}}$. Namely, $\mathcal{R}$ is a solution of the Yang-Baxter equation iff

$$\begin{pmatrix} -a_0 + \alpha & -\alpha - s \\ \alpha - s & -a_0 - \alpha \end{pmatrix} \qquad (46)$$

is a solution of the Yang-Baxter equation for $\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$.

Theorem 2.3 *Let*

$$\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, A^* = -A, D^* = -D$$

be a solution of the Yang-Baxter equation for $\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$. It has the form (46) iff

$$A + B = B^* + D. \qquad (47)$$

Under this condition

$$\alpha = \frac{B^* - B}{2}, \quad a_0 = -\frac{A + D}{2}, \quad s = -\frac{B + B^*}{2}. \qquad (48)$$

Notice that we again come to the condition (18).

Remark. It may be showed that it is not necessary to impose the condition $B \in End\left(\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}\right)$ a priori. Actually, this condition follows from the generalization of theorem 2.2 for a direct sum of two copies of an affine Lie algebra, because $\alpha \in \overset{\circ}{\mathbf{g}} \wedge \overset{\circ}{\mathbf{g}}$, $s \in \overset{\circ}{\mathbf{g}} \otimes \overset{\circ}{\mathbf{g}}$ for every solution of the modified Yang-Baxter equation for $\mathbf{g} \oplus \mathbf{g}$ of the form (40).

Unfortunately, condition (47) is not stable under the automorphisms of $\overset{\circ}{\mathbf{g}}$. So we cannot use the Belavin-Drinfeld theorem to classify all regularizations. But this theorem gives a possibility to construct sufficiently general examples of such regularizations. These examples will be presented in the next section. In the $\widehat{sl}(2)$ case we shall able to classify all regularizations.

3 Main examples of regularizations

Now we are ready to discuss examples of regularizations using the results of the previous sections. We shall consider affine Lie algebras of type $X_N^{(1)}$ and $X_N^{(2)}$ in the loop realization. We shall describe the corresponding lattice quantum algebras and their Casimir elements. In the $\widehat{sl}(2)$ case we shall explain the Algebraic Bethe Ansatz construction for such algebras.

Example 3.1 Nontwisted loop algebras.

The first example is connected with the r-matrix of the double of a finite dimensional Lie algebra $\overset{\circ}{\mathbf{g}}$ equipped with the structure of a quasitriangular Lie bialgebra. Let $\mathbf{g}$ be an affine Lie algebra of type $X_N^{(1)}$, $\overset{\circ}{\mathbf{g}}$ the corresponding finite-dimensional Lie algebra. To apply theorem 2.3 consider the r-matrix of its double $\overset{\circ}{\mathbf{d}}=\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$; we have

$$r = \begin{pmatrix} \alpha & -2\alpha_+ \\ 2\alpha_- & -\alpha \end{pmatrix}, \tag{49}$$

where α is some solution of the modified Yang-Baxter equation for $\overset{\circ}{\mathbf{g}}$ and $\alpha_\pm = \frac{1}{2}(\alpha \pm I)$. ($I$ is the identity operator in $\overset{\circ}{\mathbf{g}}$; its kernel is the Casimir element t.) According to theorem 2.3, in this case one gets:

$$s = I, a_0 = 0. \tag{50}$$

In this case $r = a + I$ is the rational r-matrix for $\mathbf{g}$. We choose for $\mathbf{g}$ the non-twisted loop realization [2]. We remind that in this realization $\mathbf{g} = \overset{\circ}{\mathbf{g}} \otimes \mathbf{C}[\lambda, \lambda^{-1}]$, and the invariant bilinear form is given by

$$\langle X(\lambda), Y(\lambda) \rangle = Res \; tr \left(X(\lambda) Y(\lambda) \right) \frac{d\lambda}{\lambda}, \tag{51}$$

where tr is an invariant bilinear form on $\overset{\circ}{\mathbf{g}}$.
The kernel of a in this realization is:

$$a(\lambda, \mu) = -t \frac{\lambda + \mu}{\lambda - \mu}, \tag{52}$$

where we identify $\mathbf{g} \otimes \mathbf{g}$ with $\overset{\circ}{\mathbf{g}} \otimes \overset{\circ}{\mathbf{g}} \otimes \mathbf{C}[\lambda, \lambda^{-1}] \otimes \mathbf{C}[\mu, \mu^{-1}]$. Thus we have the following formulas for $-a \pm \alpha$:

$$
\begin{aligned}
-a + \alpha &= \frac{\lambda}{\lambda-\mu} 2\alpha_+ - \frac{\mu}{\lambda-\mu} 2\alpha_-, \\
-a - \alpha &= \frac{\mu}{\lambda-\mu} 2\alpha_+ - \frac{\lambda}{\lambda-\mu} 2\alpha_-.
\end{aligned}
\tag{53}
$$

Let $\mathcal{R}_\pm$ be the finite-dimensional quantum R-matrix in the fundamental representation, which corresponds to $2\alpha_\pm$ after quantization. In the classical limit

$$
\mathcal{R}_\pm = I + 2\alpha_\pm h + o(h),
\tag{54}
$$

where h is the deformation parameter.
We have:

$$
\mathcal{R}_- = P\left(\mathcal{R}_+^{-1}\right),
\tag{55}
$$

where P is the permutation operator in the tensor square.

Using these data we may construct the quantum R-matrices corresponding to $-a \pm \alpha$. If we denote the quantum R-matrix corresponding to $-a - \alpha$ by $\mathcal{R}(\lambda, \mu)$, then

$$
\mathcal{R}(\lambda, \mu) = \frac{\lambda}{\lambda-\mu} \mathcal{R}_-^{-1} - \frac{\mu}{\lambda-\mu} \mathcal{R}_+^{-1},
\tag{56}
$$

and the quantum R-matrix $\mathcal{R}(\lambda, \mu)^T$ corresponds to $-a + \alpha$:

$$
\mathcal{R}(\lambda, \mu)^T = \frac{\lambda}{\lambda-\mu} \mathcal{R}_+ - \frac{\mu}{\lambda-\mu} \mathcal{R}_-,
\tag{57}
$$

here T is the conjugation with respect to the scalar product tr.
Finally, we have the quantum R-matrix on the square of $\mathbf{g}$:

$$
\mathcal{R}_q = \begin{pmatrix} \mathcal{R}(\lambda, \mu)^T & \mathcal{R}_+^{-1} \\ \mathcal{R}_- & \mathcal{R}(\lambda, \mu) \end{pmatrix}.
\tag{58}
$$

According to theorem 1.2 one can get the relations in the quantum lattice algebra $Fun_q^{\mathcal{R}}(\mathsf{G}_\tau)$, which gives a lattice approximation of the continuous system:

$$\mathcal{R}(\lambda,\mu)^T L_1^n(\lambda) L_2^n(\mu) = L_2^n(\mu) L_1^n(\lambda) \mathcal{R}(\lambda,\mu),$$
$$L_1^n(\lambda) \mathcal{R}_-^{-1} L_2^{n+1}(\mu) = L_2^{n+1}(\mu) L_1^n(\lambda), \tag{59}$$
$$L^n(\lambda) \in Fun_q^{\mathcal{R}}(\mathsf{G}_\tau) \otimes End(V),$$

here V is the fundamental representation space.

From theorem 1.3 we get the following commutation relations for the monodromy matrix:

$$\mathcal{R}(\lambda,\mu)^T M_1(\lambda) \mathcal{R}_-^{-1} M_2(\mu) = M_2(\mu) \mathcal{R}_+ M_1(\lambda) \mathcal{R}(\lambda,\mu). \tag{60}$$

The algebra (59) is connected with the Lattice Kac-Moody algebra $\mathcal{A}_{LC}$ [4]. Namely, the algebra (59) admits a family of representation for which there exists the limit

$$L^n(\lambda) \xrightarrow[\lambda\to0]{} \lambda^{-k_n} \overset{\circ}{L}{}^n + o\left(\lambda^{-k_n}\right), k_n \in \mathbf{Z}. \tag{61}$$

From (56), (57) we have the asymptotic conditions for $\mathcal{R}(\lambda,\mu), \mathcal{R}(\lambda,\mu)^T$:

$$\mathcal{R}(\lambda,\mu) \xrightarrow[\mu\to0]{} \mathcal{R}_-^{-1},$$
$$\mathcal{R}(\lambda,\mu)^T \xrightarrow[\mu\to0]{} \mathcal{R}_+. \tag{62}$$

Using (59),(62), one can get the relations for $\overset{\circ}{L}{}^n$:

$$\overset{\circ}{L}{}_2^n \overset{\circ}{L}{}_1^n = \mathcal{R}_+ \overset{\circ}{L}{}_1^n \overset{\circ}{L}{}_2^n \mathcal{R}_-$$
$$\overset{\circ}{L}{}_1^n \mathcal{R}_-^{-1} \overset{\circ}{L}{}_2^{n+1} = \overset{\circ}{L}{}_2^{n+1} \overset{\circ}{L}{}_1^n \tag{63}$$

These are the relations in the Lattice Kac-Moody algebra $\mathcal{A}_{LC}$. If $\alpha = \sum_{\alpha\in\overset{\circ}{\triangle}_+} e_\alpha \wedge e_{-\alpha}$, then the monodromy matrix for this algebra

$$\overset{\circ}{M} = \overset{\circ}{L}{}^1 \cdot \ldots \cdot \overset{\circ}{L}{}^N \tag{64}$$

satisfies the commutation relations for the quantum group $U_q\left(\overset{\circ}{\mathbf{g}}\right)$:

$$\mathcal{R}_+ \overset{\circ}{M}_1 \mathcal{R}_-^{-1} \overset{\circ}{M}_2 = \overset{\circ}{M}_2 \mathcal{R}_+ \overset{\circ}{M}_1 \mathcal{R}_-^{-1}. \tag{65}$$

This construction is useful for the computation of the center of the algebra (59).

Theorem 3.1 *For generic q and N odd the center of the algebra (59) contains the elements*

$$C_k = tr_q \left(\overset{\circ}{M} \right)^k = tr\ q^{2\rho} \left(\overset{\circ}{M} \right)^k, k = 1, \ldots, rk\ \overset{\circ}{\mathbf{g}}\ . \qquad (66)$$

(Here ρ is half the sum of positive roots in $U_q \left(\overset{\circ}{\mathbf{g}} \right)$.)

It is natural to expect that in an appropriate topology these elements generate the center of the algebra (59).

The proof of this theorem is complete similar to the proof of such theorem for the algebra $\mathcal{A}_{LC}$ [4].

From this theorem one can deduce that the center of the algebra (59) coincides with the center of $\mathcal{A}_{LC}$ [4] and with the center of $U_q \left(\overset{\circ}{\mathbf{g}} \right)$. Thus we have

Corollary 3.1 *The extensions*

$$centU_q \left(\overset{\circ}{\mathbf{g}} \right) \subset U_q \left(\overset{\circ}{\mathbf{g}} \right) \subset \mathcal{A}_{LC} \subset Fun_q^{\mathcal{R}} (\mathsf{G}_\tau)$$

are central.

Example 3.2 Twisted loop algebras.

We are leaving for a short time our first example to consider the second one. Let us consider an affine Lie algebra $\mathbf{g}$ of a type $X_N^{(r)}, r = 1, 2, 3$. Let

$$a_0 = \sum_{\alpha \in \overset{\circ}{\triangle}_+} e_\alpha \wedge e_{-\alpha}, \alpha = 0, s \in End \left(\overset{\circ}{\mathbf{h}} \right). \qquad (67)$$

It is not difficult to verify that the corresponding r-matrix of the type (46) satisfies the Yang-Baxter equation for $\overset{\circ}{\mathbf{g}} \oplus \overset{\circ}{\mathbf{g}}$. In this case $r = P_+ - P_- + s$, where $P_\pm$ are projection operators onto the opposite Borel subalgebras of $\mathbf{g}$. We shall use a twisted loop realization for $\mathbf{g}$ with invariant bilinear form (51) [2], [5]. In this realization $\mathbf{g}$ is the set of stable points of an automorphism

of an affine Lie algebra $X_{N'}^{(1)}$ in a non-twisted loop realization. Here N' does not in general coincide with N. The automorphism $\widehat{C}$ of $X_{N'}^{(1)}$ is given by

$$\widehat{C}X(\lambda) = CX\left(\lambda e^{-\frac{2\pi i}{\tau h}}\right), \tag{68}$$

where h is the Coxeter number of the algebra $X_N^{(r)}$, C is the Coxeter automorphism of $\overset{\circ}{\mathbf{g}}{}'$ which corresponds to the affine Lie algebra $X_{N'}^{(1)}$.

If $\mathbf{G} = X_N^{(1)}$, N' coincides with N and $\overset{\circ}{\mathbf{g}}{}'$ coincides with $\overset{\circ}{\mathbf{g}}$. For simplicity we restrict ourselves to this case. The automorphism C has degree h and the algebra $\overset{\circ}{\mathbf{g}}$ is decomposed into the direct sum $\overset{\circ}{\mathbf{g}} = \overset{h-1}{\underset{j=0}{\oplus}} \overset{\circ}{\mathbf{g}}_j, \overset{\circ}{\mathbf{g}}_0 = \overset{\circ}{\mathbf{h}}$, where $\overset{\circ}{\mathbf{g}}_j$ is the eigenspace of C corresponding to the eigenvalue $e^{\frac{2\pi i}{h}j}$. The Casimir element of $\overset{\circ}{\mathbf{g}}$ is decomposed into the sum $t = \overset{h-1}{\underset{j=0}{\sum}} t_j, t_j \in \overset{\circ}{\mathbf{g}}_j \otimes \overset{\circ}{\mathbf{g}}_{-j}, t_0 \in \overset{\circ}{\mathbf{h}} \otimes \overset{\circ}{\mathbf{h}}$. The kernel of the operator a in this realization is:

$$a(\lambda, \mu) = -\frac{\overset{h-1}{\underset{j=0}{\sum}} t_j\left(\lambda^{h-j}\mu^j + \lambda^j\mu^{h-j}\right)}{\lambda^h - \mu^h}. \tag{69}$$

We suppose that $s = t_0$. Now we are ready to describe the quantization. Let $\mathcal{R}(\lambda, \mu)$ be the quantum R-matrix in the fundamental representation V which corresponds to $a(\lambda, \mu)$ after quantization. Evidently, such quantum r-matrix exists because the classical r-matrix a coincides with the classical r-matrix $a + \alpha$ from example 1 if $\alpha = \underset{\alpha \in \overset{\circ}{\triangle}_+}{\sum} e_\alpha \wedge e_{-\alpha}$. For the latter classical r-matrix the corresponding quantum r-matrix exists (see example 1). Let $\mathcal{R}_0 = e^{hs}$. We have the following quantum R-matrix on the square:

$$\mathcal{R}_q = \begin{pmatrix} \mathcal{R}(\lambda, \mu)^{-1} & \mathcal{R}_0^{-1} \\ \mathcal{R}_0^{-1} & \mathcal{R}(\lambda, \mu)^{-1} \end{pmatrix}. \tag{70}$$

In the standard way we built the quantum lattice algebra $Fun_q^{\mathcal{R}}(\mathsf{G}_\tau)$ which gives a lattice counterpart for our continuous system. According to theorem 1.2, the commutation relations for it have the form

$$\begin{aligned} \mathcal{R}(\lambda, \mu) L_1^n(\lambda) L_2^n(\mu) &= L_2^n(\mu) L_1^n(\lambda) \mathcal{R}(\lambda, \mu), \\ L_1^n(\lambda) \mathcal{R}_0 L_2^{n+1}(\mu) &= L_2^{n+1}(\mu) L_1^n(\lambda), \\ L^n(\lambda) &\in Fun_q^{\mathcal{R}}(\mathsf{G}_\tau) \otimes End(V). \end{aligned} \tag{71}$$

For the monodromy matrix we have the relation:

$$\mathcal{R}\left(\lambda,\mu\right)M_1\left(\lambda\right)\mathcal{R}_0 M_2\left(\mu\right)=M_2\left(\mu\right)\mathcal{R}_0 M_1\left(\lambda\right)\mathcal{R}\left(\lambda,\mu\right). \qquad (72)$$

Similarly to theorem 3.1, there exists a description of the center of the algebra (71).

Theorem 3.2 *Let*

$$M\underset{\lambda\to 0}{\longrightarrow}\lambda^{-k}\overset{\circ}{M}+o\left(\lambda^{-k}\right). \qquad (73)$$

For a generic q and N odd the center of the algebra (71) contains the elements:

$$C_k=tr\left(\overset{\circ}{M}\right)^k, k=1,\dots,rk\ \overset{\circ}{\mathbf{g}}. \qquad (74)$$

As in theorem 3.1, it is natural to expect that in an appropriate topology these elements generate the center of the algebra (71).

The algebra (71), as opposed to the algebra (59), possesses a remarkable property; namely, it admits an ultralocalization. This property is important for the construction of representations of lattice algebras. The exact statement is given by the following theorem.

Theorem 3.3 *Let ρ be the fundamental representation of the group $\overset{\circ}{H}$ corresponding to the algebra $\overset{\circ}{\mathbf{h}}$, and V the space of its fundamental representation used for the definition of the algebra (71). Then ρ acts naturally in V. Let A be the free algebra generated by the matrix coefficients of the matrices $\widehat{L}^n\in A\otimes EndV, G^n\in A\otimes\rho$, satisfying the relations:*

$$\begin{aligned}\mathcal{R}\left(\lambda,\mu\right)\widehat{L}_1^n\left(\lambda\right)\widehat{L}_2^n\left(\mu\right)&=\widehat{L}_2^n\left(\mu\right)\widehat{L}_1^n\left(\lambda\right)\mathcal{R}\left(\lambda,\mu\right),\\ G_1^n\widehat{L}_2^{n+1}\left(\mu\right)&=\mathcal{R}_0^{\frac{1}{2}}\widehat{L}_2^{n+1}\left(\mu\right)G_1^n,\\ G_1^n\widehat{L}_2^n\left(\mu\right)&=\widehat{L}_2^n\left(\mu\right)G_1^n\mathcal{R}_0^{\frac{1}{2}}.\end{aligned} \qquad (75)$$

Then there exist the homomorphism of the algebras:

$$\begin{aligned}Fun_q^{\mathcal{R}}\left(\mathsf{G}_\tau\right)&\to A,\\ L^n&\mapsto G^{n-1}\widehat{L}^n G^{n-1},\\ M&\mapsto G^N\widehat{M}G^{N-1};\\ here\ \widehat{M}&=\widehat{L}^1\cdot\ldots\cdot\widehat{L}^N.\end{aligned} \qquad (76)$$

Remark. This homomorphism has the form of a lattice gauge transformation: the generators L^n of the nonultralocal algebra $Fun_q^{\mathcal{R}}(\mathsf{G}_\tau)$ are connected by a lattice gauge transformation with the generators $\widehat{L}^n$ satisfying ultralocal relations. Therefore this homomorphism is called ultralocalization. The monodromy matrices M and $\widehat{M}$ are conjugate and hence nonultralocality is equivalent to including some quasiperiodic boundary conditions. This is natural in the spirit of the operator extension theory used in the previous section. In particular examples it is easier to construct representations of the algebra A than those of the algebra (71). It is the main motivation for the definition of the algebra A.

The proof of this theorem consists in a straightforward check of the relations (71) for the generators $G^{n-1}\widehat{L}^n G^{n^{-1}}$. A similar theorem on ultralocalization exists for the algebra $\mathcal{A}_{LC}$ [13].

Now we are ready to describe all regularized r-matrices and their regularizations for the algebra $A_1^{(1)}$. According to the Belavin-Drinfeld theorem for finite-dimensional semisimple Lie algebras [5], we may construct all solutions of the modified Yang-Baxter equation on the square of corresponding finite-dimensional Lie algebra $\overset{\circ}{\mathsf{g}} = sl(2)$. Modulo some freedom in the choice of the Cartan components of R-matrix on the skew-diagonal there exist only two solutions of the Yang-Baxter equation on $sl(2) \oplus sl(2)$ considered in examples 1 and 2 in this section. Moreover, the r-matrix of the double (49) in example 1 is connected with the unique structure of a bialgebra on $sl(2)$ for which

$$\alpha = \sum_{\alpha \in \overset{\circ}{\triangle}_+} e_\alpha \wedge e_{-\alpha}. \tag{77}$$

For the algebra $A_1^{(1)}$ the Coxeter number h equals 2, and the Coxeter automorphism has the form:

$$CX = DXD^{-1},$$
$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{78}$$

We summarize all results for the algebra $A_1^{(1)}$ in the following theorem. We formulate at once the quantum version of all formulas. The reader may reproduce the corresponding classical formulas by a quasiclassical limit.

Theorem 3.4 *For the algebra $A_1^{(1)}$ there exists only two nonultralocal Lie-Poisson brackets of the type (31) which may be regularized in the way (37). Namely, let $b_\pm$ be the opposite Borel subalgebras in $A_1^{(1)}$ and $\overset{\circ}{b}_\pm$ the opposite Borel subalgebras in $sl\,(2)$. Then the regularized r-matrices and the corresponding lattices algebras are:*

1. *$r = P_{b_+ \backslash \overset{\circ}{b}_+} - P_{b_- \backslash \overset{\circ}{b}_-} + I$ is the regularized r-matrix, here $P_{b_+ \backslash \overset{\circ}{b}_+}, P_{b_- \backslash \overset{\circ}{b}_-}$ are projectors onto the corresponding subspaces, I is the identity operator in $sl\,(2)$. It is precisely the rational r-matrix. If we use the non-twisted current realization $A_1^{(1)} = sl\,(2) \otimes \mathbf{C}\,[\lambda, \lambda^{-1}]$ and the scalar product (51), its kernel is given by*

$$r\,(\lambda, \mu) = -\frac{\mu t}{\lambda - \mu}, \tag{79}$$

   where t is the Casimir element for $sl\,(2)$.

   The relations in the corresponding lattice algebra are given by formulas (59),where for the fundamental representation $A_1^{(1)}$ in $\mathbf{C}^2\,[\lambda, \lambda^{-1}]$:

$$\mathcal{R}_+ = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$
$$\mathcal{R}_- = P\left(\mathcal{R}_+^{-1}\right), \tag{80}$$

$$\mathcal{R}\,(\lambda, \mu) = \begin{pmatrix} \frac{\lambda q - \mu q^{-1}}{\lambda - \mu} & 0 & 0 & 0 \\ 0 & 1 & \frac{\mu}{\lambda - \mu}(q - q^{-1}) & 0 \\ 0 & \frac{\lambda}{\lambda - \mu}(q - q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & \frac{\lambda q - \mu q^{-1}}{\lambda - \mu} \end{pmatrix} \tag{81}$$

   .

2. *$r = P_{b_+} - P_{b_-} + \xi P_{\overset{\circ}{h}}$ is the regularized r-matrix, $P_{\overset{\circ}{h}}$ is the projection operator onto the Cartan subalgebra of $sl\,(2)$. For simplicity we consider only the case $\xi = 1$. Fix the twisted current realization*

$$A_1^{(1)} = \left\{ X\left(\lambda\right) \in sl\left(2\right) \otimes \mathbf{C}\left[\lambda, \lambda^{-1}\right] : X\left(-\lambda\right) = DX\left(\lambda\right)D^{-1} \right\} \quad (82)$$

and the scalar product (51). Then the kernel of r is given by

$$r\left(\lambda, \mu\right) = -\frac{\mu^2 t_0}{\lambda^2 - \mu^2} - \frac{\lambda \mu t_1}{\lambda^2 - \mu^2}, \quad (83)$$

where t_0 is the $\overset{\circ}{\mathbf{h}}$ – component of the $sl\left(2\right)$ Casimir element and $t_1 \in \overset{\circ}{\mathbf{g}}_1 \otimes \overset{\circ}{\mathbf{g}}_{-1}, \overset{\circ}{\mathbf{g}}_1 \in \left\{ X \in sl\left(2\right), DXD^{-1} = -X \right\}$. The relations in the corresponding lattice algebra are given by the formulas (71), where for the fundamental representation $A_1^{(1)}$ in $\mathbf{C}^2\left[\lambda, \lambda^{-1}\right]$:

$$\mathcal{R}_0 = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad (84)$$

$$\mathcal{R}\left(\lambda, \mu\right) = \begin{pmatrix} \frac{\lambda^2 q - \mu^2 q^{-1}}{\lambda^2 - \mu^2} & 0 & 0 & 0 \\ 0 & 1 & \frac{2\lambda\mu}{\lambda^2 - \mu^2}\left(q - q^{-1}\right) & 0 \\ 0 & \frac{2\lambda\mu}{\lambda^2 - \mu^2}\left(q - q^{-1}\right) & 1 & 0 \\ 0 & 0 & 0 & \frac{\lambda^2 q - \mu^2 q^{-1}}{\lambda^2 - \mu^2} \end{pmatrix}. \quad (85)$$

Remark.

The quantum R-matrices (81) and (85) are in fact the ordinary trigonometric quantum R-matrices in different realizations. To see this just notice that the classical r-matrices $a + \alpha$ from the example 1 (see (50),(45),(77)) and a from the example 2 (see (45), (67) coincide.

We conclude this section with the Bethe-Ansatz construction for algebras with relations (59), (71) in the $A_1^{(1)}$ case.

We start with example 1.The 'generating function' producing an infinite series of quantum commuting integrals of motion [8], [9] is

$$tr_q M\left(\lambda\right) = qA\left(\lambda\right) + q^{-1}D\left(\lambda\right),$$
$$\text{where } M\left(\lambda\right) = \begin{pmatrix} A\left(\lambda\right) & B\left(\lambda\right) \\ C\left(\lambda\right) & D\left(\lambda\right) \end{pmatrix} \tag{86}$$
$$\text{and } \left[tr_q M\left(\lambda\right), tr_q M\left(\mu\right)\right] = 0.$$

We suppose that there exists a reference state Ω:

$$A\left(\lambda\right)\Omega = a\left(\lambda\right)\Omega, D\left(\lambda\right)\Omega = d\left(\lambda\right)\Omega, C\left(\lambda\right)\Omega = 0, B\left(\lambda\right)\Omega \neq 0. \tag{87}$$

We shall try to seek a representation of the algebra (60) in which $tr_q M\left(\lambda\right)$ is diagonal. According to the standard Bethe Ansatz technique we try to find the eigenvectors of $tr_q M\left(\lambda\right)$ in the form:

$$\psi\left(\lambda_1, \ldots, \lambda_n\right) = B\left(\lambda_1\right)\ldots B\left(\lambda_n\right)\Omega. \tag{88}$$

The relations between $A\left(\lambda\right), B\left(\lambda\right), C\left(\lambda\right), D\left(\lambda\right)$ which are essential to calculate the spectrum of $tr_q M\left(\lambda\right)$ are:

$$[B\left(\lambda\right), B\left(\mu\right)] = 0,$$
$$A\left(\lambda\right)B\left(\mu\right) = \frac{\mu - \lambda q^{-2}}{\mu - \lambda}B\left(\mu\right)A\left(\lambda\right) - \frac{\lambda}{\mu - \lambda}\left(1 - q^{-2}\right)B\left(\lambda\right)A\left(\mu\right) -$$
$$- \left(1 - q^{-2}\right)B\left(\lambda\right)D\left(\mu\right), \tag{89}$$
$$D\left(\lambda\right)B\left(\mu\right) = \frac{\mu}{\mu - \lambda}\left(q^2 - 1\right)B\left(\lambda\right)D\left(\mu\right) + \frac{\mu - \lambda q^2}{\mu - \lambda}B\left(\mu\right)D\left(\lambda\right).$$

The vector $\psi\left(\lambda_1, \ldots, \lambda_n\right)$ will be an eigenvector of $tr_q M\left(\lambda\right)$ with the eigenvalue

$$qa\left(\lambda\right)\prod_{i=1}^{n}\frac{\lambda q^{-2} - \lambda_i}{\lambda - \lambda_i} + q^{-1}d\left(\lambda\right)\prod_{i=1}^{n}\frac{\lambda q^2 - \lambda_i}{\lambda - \lambda_i} \tag{90}$$

iff λ_i satisfy the equations [8],[9]:

$$\frac{d\left(\lambda_j\right)}{a\left(\lambda_j\right)} = \prod_{i \neq j}\frac{\lambda_j q^{-2} - \lambda_i}{\lambda_j q^2 - \lambda_i}. \tag{91}$$

Similarly, in example 2 the element

$$trM\left(\lambda\right) = A\left(\lambda\right) + D\left(\lambda\right),$$
$$\text{where } M\left(\lambda\right) = \begin{pmatrix} A\left(\lambda\right) & B\left(\lambda\right) \\ C\left(\lambda\right) & D\left(\lambda\right) \end{pmatrix} \tag{92}$$

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produces an infinite series of commuting conservation laws. Under the assumption (87) we look for the eigenvectors of $trM(\lambda)$ which have the form (88). The essential commutation relations for this procedure are (72):

$$[B(\lambda), B(\mu)] = 0,$$
$$A(\lambda) B(\mu) = \frac{\mu^2 - \lambda^2 q^{-2}}{\mu^2 - \lambda^2} B(\mu) A(\lambda) - \frac{2\lambda\mu}{\mu^2 - \lambda^2} (1 - q^{-2}) B(\lambda) A(\mu),$$
$$D(\lambda) B(\mu) = \frac{2\lambda\mu}{\mu^2 - \lambda^2} (q^2 - 1) B(\lambda) D(\mu) + \frac{\mu^2 - \lambda^2 q^2}{\mu^2 - \lambda^2} B(\mu) D(\lambda). \tag{93}$$

The vector $\psi(\lambda_1, \ldots, \lambda_n)$ is an eigenvector of $trM(\lambda)$ with the eigenvalue

$$a(\lambda) \prod_{i=1}^{n} \frac{\lambda^2 q^{-2} - \lambda_i^2}{\lambda^2 - \lambda_i^2} + d(\lambda) \prod_{i=1}^{n} \frac{\lambda^2 q^2 - \lambda_i^2}{\lambda^2 - \lambda_i^2} \tag{94}$$

iff λ_i satisfy the equations:

$$q^2 \frac{d(\lambda_j)}{a(\lambda_j)} = \prod_{i \neq j} \frac{\lambda_j^2 q^{-2} - \lambda_i^2}{\lambda_j^2 q^2 - \lambda_i^2}. \tag{95}$$

We omit the standard Bethe Ansatz calculations leading to these formulas (90), (91), (94), (95) [16]. It is easy to verify that the modification of the commutation relations (89), (93) (as compared to the standard XXZ relations) only weakly influences the Bethe Ansatz construction, so that the principal idea [16] may be applied as before.

4 Application to the nonlinear sigma model

We recall some facts about chiral fields with values in Riemannian symmetric spaces. Let $(\mathbf{k}, \mathbf{p})$ be a Riemannian symmetric pair for a semisimple finite-dimensional Lie algebra $\overset{\circ}{\mathbf{g}}$ [15]. It means that $\overset{\circ}{\mathbf{g}} = \mathbf{k} \dotplus \mathbf{p}$ as a linear space, and

$$[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}, [\mathbf{k}, \mathbf{p}] \subset \mathbf{p}, [\mathbf{p}, \mathbf{p}] \subset \mathbf{k}. \tag{96}$$

so that $\mathbf{k}$ is a subalgebra in $\overset{\circ}{\mathbf{g}}$. We suppose that the decomposition $\overset{\circ}{\mathbf{g}} = \mathbf{k} \dotplus \mathbf{p}$ is orthogonal with respect to the standard scalar product on $\overset{\circ}{\mathbf{g}}$, so that the projectors $P_{\mathbf{p}}$ and $P_{\mathbf{k}}$ onto the subspaces $\mathbf{p}$ and $\mathbf{k}$ are orthogonal. Let $\widetilde{\overset{\circ}{\mathbf{g}}} = C^\infty \left(S^1, \overset{\circ}{\mathbf{g}} \right)$ and let us introduce the left currents $l_x, l_t \in \widetilde{\overset{\circ}{\mathbf{g}}}$,

$$l_x = g^{-1}\partial_x g, l_t = g^{-1}\partial_t g, g \in C^\infty\left(S^1, \overset{\circ}{G}\right) = \widetilde{\overset{\circ}{G}}, \tag{97}$$

where $\overset{\circ}{G}$ is a Lie group corresponding to $\overset{\circ}{\mathbf{g}}$. Let A_μ, B_μ be the two projections of the current, :

$$A_\mu = P_{\mathbf{k}} l_\mu, B_\mu = P_{\mathbf{p}} l_\mu. \tag{98}$$

In these notations the action of the nonlinear sigma model has the form [14]:

$$S\left(g\right) = \tfrac{1}{2}\int tr\left(B_x B_x - B_t B_t\right) dx dt,$$
$$S\left(g\right) \in Fun\left(\overset{\circ}{\widetilde{G}}\right). \tag{99}$$

The action functional is unchanged under the right gauge action of the group $\widetilde{K} = C^\infty\left(S^1, K\right)$:

$$g \mapsto gk, k \in \widetilde{K}, g \in \widetilde{\overset{\circ}{G}}, \tag{100}$$

so that this is a well defined function on the symmetric space $\widetilde{\overset{\circ}{G}}/\widetilde{K}$. The equations of motion which follow from the action (99) are:

$$\begin{aligned}
\partial_t B_t + [A_t, B_t] &= \partial_x B_x + [A_x, B_x], \\
\partial_x A_t - \partial_t A_x + [A_x, A_t] + [B_x, B_t] &= 0, \\
\partial_x B_t - \partial_t B_x + [A_x, B_t] - [A_t, B_x] &= 0.
\end{aligned} \tag{101}$$

The two last equations are zero curvature conditions which serve to restore the group variable $g \in \widetilde{\overset{\circ}{G}}/\widetilde{K}$ from A_μ, B_μ.

For this model there exists a Lax pair:

$$\begin{aligned}
L &= -\left(A_x + \tfrac{\lambda}{2}\left(B_x + B_t\right) + \tfrac{1}{2\lambda}\left(B_x - B_t\right)\right), \\
T &= -\left(A_t + \tfrac{\lambda}{2}\left(B_x + B_t\right) - \tfrac{1}{2\lambda}\left(B_x - B_t\right)\right).
\end{aligned} \tag{102}$$

The equations of motion (101) are expressed as the zero curvature condition:

$$[\partial_x - L, \partial_t - T] = 0. \tag{103}$$

Now we define a natural Poisson structure connected with the Lax pair (102). Let $\mathbf{g} = \overset{\circ}{\mathbf{g}} \otimes \mathbf{C}\left[\lambda, \lambda^{-1}\right]$ be the current Lie algebra with the scalar product (51). Let $\mathbf{g}^\sigma$ be the twisted current Lie algebra which corresponds to the Cartan automorphism σ associated with $(\mathbf{k}, \mathbf{p})$:

$$\sigma \mid_{\mathbf{k}} = id, \sigma \mid_{\mathbf{p}} = -id. \tag{104}$$

Thus

$$\mathbf{g}^\sigma = \left\{ X\left(\lambda\right) \in \mathbf{g} : \sigma X\left(-\lambda\right) = X\left(\lambda\right) \right\}. \tag{105}$$

For our model we may choose the standard r-matrix Lie-Poisson structure bracket (31) using the r-matrix

$$r = P_+ - P_-; \tag{106}$$

where P_- is the projection operator onto the negative part of a Laurent series and P_+ is the complementary projection operator . The projection operator P_- has the kernel:

$$P_-\left(\lambda, \mu\right) = t_A \frac{\mu^2}{\lambda^2 - \mu^2} + t_B \frac{\lambda\mu}{\lambda^2 - \mu^2}, \tag{107}$$

where $t_A = \left(P_{\mathbf{k}} \otimes P_{\mathbf{k}}\right) t, t_B = \left(P_{\mathbf{p}} \otimes P_{\mathbf{p}}\right) t$ are the $\mathbf{k}-$ and $\mathbf{p}-$ components of the Casimir element, respectively. The r-matrix (106) has a symmetric part with the kernel:

$$s\left(\lambda, \mu\right) = t_A. \tag{108}$$

Example 4.1 The principal chiral field.

Let $\overset{\circ}{\mathbf{g}} = \mathbf{a} \oplus \mathbf{a}$ be the direct sum of two copies of a simple Lie algebra $\mathbf{a}$. Let $\mathbf{k}$ be the diagonal subalgebra in $\overset{\circ}{\mathbf{g}}$ and $\mathbf{p}$ be the anti-diagonal subspace:

$$\begin{aligned} \mathbf{k} &= \left\{ \left(x, x\right), x \in \mathbf{a} \right\}, \\ \mathbf{p} &= \left\{ \left(x, -x\right), x \in \mathbf{a} \right\}, \end{aligned} \tag{109}$$
so that corresponding automorphism σ is $\sigma\left(x, y\right) = \left(y, x\right).$

This is a Riemannian symmetric pair. The symmetric space for the corresponding current group is $\widetilde{A} \times \widetilde{A}/\widetilde{A}$, where $\widetilde{A}$ is the current group of the Lie group corresponding to the Lie algebra **a**. In this case $\widetilde{A} \simeq \widetilde{P}$ and acts on $\widetilde{A} \times \widetilde{A}$ according to (100):

$$\widetilde{A} \times \left(\widetilde{A} \times \widetilde{A}\right) \to \widetilde{A} \times \widetilde{A},$$
$$g \circ (g_1, g_2) \mapsto (g_1 g, g_2 g)$$

So in this case the action (99) is a well-defined function on $\widetilde{A}$. If we define the projection

$$\pi : \widetilde{A} \times \widetilde{A} \to \widetilde{A},$$
$$\pi : (g_1, g_2) \mapsto g_1 g_2^{-1} = g \tag{110}$$

onto the quotient space $\widetilde{A} \simeq \widetilde{A} \times \widetilde{A}/\widetilde{A}$ and define the currents

$$l_\mu^k = g_k^{-1} \partial_\mu g_k, k = 1, 2, l_\mu = g^{-1} \partial_\mu g, \tag{111}$$

then the variables A_μ and B_μ have the form:

$$A_\mu = \left(\tfrac{1}{2}\left(l_\mu^1 + l_\mu^2\right), \tfrac{1}{2}\left(l_\mu^1 + l_\mu^2\right)\right),$$
$$B_\mu = \left(\tfrac{1}{2}\left(l_\mu^1 - l_\mu^2\right), -\tfrac{1}{2}\left(l_\mu^1 - l_\mu^2\right)\right), \tag{112}$$

and the action (99) takes the form:

$$S\left(g\right) = \int \left(l_x l_x - l_t l_t\right) dx dt. \tag{113}$$

This is the action of the principal chiral field model on $\widetilde{A}$, so that we deal with a realization of this model on the symmetric space $\widetilde{A} \times \widetilde{A}/\widetilde{A}$ .The Lax operator (102) is a direct sum of the two ones:

$$L\left(\lambda\right) = \left(L^1\left(\lambda\right), L^1\left(-\lambda\right)\right), \tag{114}$$

and the r-matrix (106) is a direct sum of a two copies of the rational r-matrix:

$$r = \begin{pmatrix} P_+ - P_- & 0 \\ 0 & P_+ - P_- \end{pmatrix}. \tag{115}$$

Let us return to the situation of example 3.1. Our model is a direct sum of two copies of this example. The quantum lattice algebra is a direct sum

of two copies of the algebra (59), where L^n is a quantum version of a lattice approximation of the Lax operator $L^1(\lambda)$. If

$$M(\lambda) = \left(M^1(\lambda), M^1(-\lambda)\right) \tag{116}$$

is the quantum monodromy matrix for our model, then $M^1(\lambda)$ satisfies the relation (60). If $\mathbf{a} = sl(2)$ then the generating function for the integrals of motion is:

$$tr_q M^1(\lambda) + tr_q M^1(-\lambda), \tag{117}$$

and we may apply the technique of the modified Bethe Ansatz developed in section 3 to calculate the spectrum.

Example 4.2 The n-field.

Let $\overset{\circ}{\mathbf{g}} = su(2)$ and let $\mathbf{k} = \overset{\circ}{\mathbf{h}}$ be the Cartan subalgebra of $su(2)$, and $\mathbf{p}$ the complementary subspace. This is a Riemannian symmetric pair and the corresponding automorphism σ is given by (cf. 78):

$$\sigma X = DXD^{-1}, X \in su(2), D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{118}$$

Let

$$A_x = \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix}, B_x = \begin{pmatrix} 0 & A_1 + A_{-1} \\ -\left(\overline{A}_1 + \overline{A}_{-1}\right) & 0 \end{pmatrix},$$
$$B_t = \begin{pmatrix} 0 & A_1 - A_{-1} \\ -\left(\overline{A}_1 - \overline{A}_{-1}\right) & 0 \end{pmatrix}, \tag{119}$$

then in this case the Lax operator (102) has the form:

$$L = -\begin{pmatrix} ia & \lambda A_1 + \frac{1}{\lambda} A_{-1} \\ -\lambda \overline{A}_1 - \frac{1}{\lambda} \overline{A}_{-1} & -ia \end{pmatrix}. \tag{120}$$

Since the automorphism σ coincides with the Coxeter automorphism (78) for $sl(2)$, the twisted affine Lie algebra $\mathbf{g}^\sigma$ (105) coincides with the compact real form of the twisted realization of the algebra $A_1^{(1)}$. To define a Poisson

structure we may use the r-matrix (106) which in this case is precisely the r-matrix from example 3.1 So that we are returning to the situation investigated in this example. The lattice quantum algebra is given by theorem 3.4, part 2, where L^n is a quantum version of a lattice approximation of the Lax operator (120). According to theorem 3.3 for this algebra there exists an ultralocalization. Let

$$
\begin{aligned}
\widehat{L}^n(\lambda) &= - \begin{pmatrix} iA_0^n & \lambda A_1^n + \frac{1}{\lambda} A_{-1}^n \\ -\lambda A_1^{n*} - \frac{1}{\lambda} A_{-1}^{n*} & iA_0^{n*} \end{pmatrix}, \\
G^n &= \begin{pmatrix} a_n & 0 \\ 0 & a_n^* \end{pmatrix},
\end{aligned}
\tag{121}
$$

where we use the notations of theorem 3.3 and $*$ is the anti-involution in this algebra (it is supposed that $\mid q \mid = 1$). We shall give an explicit realization of this algebra via the canonical Weyl pairs

$$
\begin{aligned}
U_i^n V_i^n &= q V_i^n U_i^n, i = 1, 2; n = 1, \ldots, N, \\
U_i^{n*} &= U_i^{n^{-1}}, V_i^{n*} = V_i^{n^{-1}}.
\end{aligned}
\tag{122}
$$

We omit the complicated calculations and give only the result:

$$
\begin{aligned}
A_0^n &= \left(1 - q^{-1} V_1^{n^{-2}}\right) U_1^n U_2^n, \\
A_1^n &= V_1^n U_2^n, A_{-1}^n = V_1^{n^{-1}} U_2^n, \\
a_n &= V_1^{n^{-\frac{1}{2}}} V_2^{n^{\frac{1}{4}}} V_2^{n+1^{-\frac{1}{4}}}.
\end{aligned}
\tag{123}
$$

The Lax operator $\widehat{L}^n(\lambda)$ coincides up to an unessential factor with the one of the Lattice Sine-Gordon model [9]:

$$
\widehat{L}^n = L_{LSG}^n \otimes U_2^n.
\tag{124}
$$

Since U_2^n is the shift operator in the standard Weyl representation of the algebra (122), we may choose the reference state in the form:

$$
\begin{aligned}
\Omega &= \Omega_{LSG} \otimes \mathbf{1}, \\
\mathbf{1} &= \underbrace{1 \otimes \ldots \otimes 1}_{N},
\end{aligned}
\tag{125}
$$

where Ω_{LSG} is the reference state of the Lattice Sine-Gordon model. U_2^n acts trivially on 1, so that the spectrum of our model coincides with the one for the Lattice Sine-Gordon model. Here we are using the gauge equivalence

of these two models and the connection between generating functions for the integrals of motion:

$$
\begin{aligned}
trM &= q^{\frac{1}{2}} tr\widehat{M}, \\
M &= L^1 \cdot \ldots \cdot L^N, \\
\widehat{M} &= \widehat{L}^1 \cdot \ldots \cdot \widehat{L}^N.
\end{aligned}
\tag{126}
$$

5 Conclusion

In this section we formulate some problems related to the subject of this paper.

The first problem is connected with a generalization of our construction of nonultralocal algebras to an arbitrary graph. It is not difficult to construct a nonultralocal algebra on a graph using a solution of the Yang-Baxter equation on the square of some Lie algebra and the 'polyuble construction' [11]. In the finite-dimensional case these algebras were introduced in [11] in connection with the quantization of the moduli space. The lattice algebra on a graph similar to one considered in example 1 of this paper was investigated in [3].

A more interesting problem is the regularization of the Poisson bracket relations for the monodromy   for a more wide class of scalar products on current algebras, e.g. when

$$
(X(\lambda), Y(\lambda)) = Res\ tr\,(X(\lambda)\,Y(\lambda))\,\phi(\lambda)\,d\lambda,
$$

where $\phi(\lambda)$ is some rational function. The Belavin-Drinfeld classification theorem does not allow to define regularization in the way considered in section 2, because there exist no nontrivial solutions of the Classical Yang-Baxter equation on $\mathbf{g} \oplus \mathbf{g}$ of the type (40) which are skew-symmetric with respect to this scalar product . One of the possible ways to tackle with this situation is to use Quasi-Hopf algebras [7]. This requires the study of r-matrices which do not satisfy the Yang-Baxter equation. However, the physical motivation to introduce such algebras is not clear.

Another problem is the construction of representations of nonultralocal algebras. These representations are important for the definition of reference states Ω and of the Bethe Ansatz construction. A natural way to solve this problem is the ultralocalization procedure. As we have seen, ultralocalization

does not exist for all lattice algebras. It is interesting to describe all representations of nonultralocal algebras and to choose from them those which arise from an ultralocalization procedure. For finite-dimensional algebras this question is connected with the anomaly problem in the quantum field theory [4].

The last problem is to construct so called fundamental Lax operator for nonultralocal algebras [9]. The first step to its solution was made in [12].

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